

CREMONA TRANSFORMATIONS AND DERIVED EQUIVALENCES OF K3 SURFACES

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ABSTRACT. We exhibit a Cremona transformation of \mathbb{P}^4 such that the base loci of the map and its inverse are birational to K3 surfaces. The two K3 surfaces are derived equivalent but not isomorphic to each other. As an application, we show that the difference of the two K3 surfaces annihilates the class of the affine line in the Grothendieck ring of varieties.

INTRODUCTION

Let X be a smooth complex projective variety that is *rational*, i.e., admits a birational map $\varrho : \mathbb{P}^r \dashrightarrow X$ where $r = \dim(X)$. The map ϱ blows up various subvarieties of \mathbb{P}^r — to what extent are these determined by X ? We can always precompose ϱ by a birational automorphism of \mathbb{P}^r (i.e., a *Cremona transformation*) so we must take this into account.

For small dimensions these subvarieties are determined by X . When $r = 1$, ϱ extends to an isomorphism; if $r = 2$, ϱ is resolved by blowing up points in \mathbb{P}^2 . The case of threefolds was analyzed by Clemens and Griffiths [CG72]. We may assume that ϱ (resp. ϱ^{-1}) is resolved by blowing up a finite number of points and nonsingular irreducible curves; let C_1, \dots, C_k (resp. D_1, \dots, D_l) denote those of positive genus. Comparing the Hodge structures on middle cohomology groups using the blow-up formula, we obtain an isomorphism of principally polarized abelian varieties:

$$J(C_1) \times \cdots \times J(C_k) \simeq J(X) \times J(D_1) \times \cdots \times J(D_l).$$

The factors are Jacobians of curves and the intermediate Jacobian of X . Principally polarized abelian varieties admit unique decompositions into irreducible factors and the Jacobian of a curve is irreducible with respect to the natural polarization. It follows that $J(X)$ can be expressed as a product of Jacobians of curves C_{i_1}, \dots, C_{i_t} , $\{i_1, \dots, i_t\} \subset \{1, \dots, k\}$, and these curves are determined up to isomorphism by the Torelli Theorem.

Therefore, we focus on fourfolds and their middle cohomology. Suppose that a smooth projective surface Σ is contained in the base locus of ϱ . The blow-up formula gives a homomorphism of Hodge structures

$$\beta : H^2(\Sigma, \mathbb{Z})(-1) \rightarrow H^4(X, \mathbb{Z});$$

can we recover Σ from its image? Keeping track of divisor classes of Σ is complicated, as they might disappear under subsequent blow-downs. Thus all we can expect to recover is the transcendental cohomology $T(\Sigma) \subset H^2(\Sigma, \mathbb{Z})$.

Mukai and Orlov [Orl97] have shown that K3 surfaces offer many examples of non-birational surfaces R and \hat{R} with $T(R) \simeq T(\hat{R})$ as integral Hodge structures. These are explained through the notion of *derived equivalence*. There are cubic fourfolds [Has16] whose Hodge structures show the trace of several derived equivalent K3 surfaces. However, these are not known to be rational. Nevertheless, this raises a question:

Question 0.1. *Let R and \hat{R} be derived equivalent K3 surfaces. Do there exist smooth projective fourfolds X , P , and \hat{P} and birational maps*

$$\varrho : P \dashrightarrow X, \quad \hat{\varrho} : \hat{P} \dashrightarrow X,$$

such that R and \hat{R} are birational to components of the base loci of ϱ and $\hat{\varrho}$ respectively, and the induced

$$\beta : H^2(R, \mathbb{Z})(-1) \rightarrow H^4(X, \mathbb{Z}), \quad \hat{\beta} : H^2(\hat{R}, \mathbb{Z})(-1) \rightarrow H^4(X, \mathbb{Z})$$

induce an isomorphism $T(R) \simeq T(\hat{R})$?

In other words, are derived equivalences of K3 surface induced by birational maps? It makes sense to start with the case where $P \simeq \hat{P} \simeq \mathbb{P}^4$. Are derived equivalences of K3 surfaces induced by Cremona transformations?

This last question may be too ambitious, as the base loci of Cremona transformations are highly constrained. According to Crauder and Katz [CK89], the Cremona transformation of \mathbb{P}^4 which can be resolved by blowing up along a smooth and irreducible surface $S \subset \mathbb{P}^4$ occurs as one of the following two cases:

- (1) S is a quintic elliptic scroll $S = \mathbb{P}_C(E)$, where C is an elliptic curve and E is a rank two vector bundle with $e = -\deg(\wedge^2 E) = -1$.
- (2) S is a degree 10 determinantal surface given by the vanishing of the 4×4 minors of a 4×5 matrix of linear forms.

Here we present an example where derived equivalences of K3 surfaces are explained through Cremona transformations, and offer further evidence that such examples are quite rare. We can explain derived equivalences among degree 12 K3 surfaces in this way; however, we do not know how to realize derived equivalences of higher degree K3 surfaces.

Our construction gives new examples of zero-divisors in the Grothendieck ring of complex algebraic varieties. The difference of each derived equivalent pair is non-vanishing in the ring and annihilated by the class of the

affine line. The first example in this direction is given by the Pfaffian-Grassmannian Calabi-Yau threefolds [Bor15]. Other examples include Calabi-Yau threefolds from Grassmannians of type G_2 [IMOU16a, Kuz16]. Kuznetsov and Shinder [KS16] have formulated general conjectures relating derived equivalence to zero-divisors in the Grothendieck ring; our example is an instance of [KS16, Conj. 1.6]. The relationship between zero-divisors in the Grothendieck ring and approaches to the rationality of cubic fourfolds are discussed in [GS14].

Section 1 presents preliminary results on Cremona transformations with singular base loci. The construction of our rational map is in Section 2 and we verify the non-trivial derived equivalence in Section 3. We apply the results to study the Grothendieck ring in Section 4. Section 5 shows these constructions do not admit obvious extensions through a generalization of the classification of Crauder and Katz; the underlying computations are also used to analyze the maps defined in Section 2.

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1. CREMONA TRANSFORMATION WITH SINGULAR BASE LOCUS

1.1. Terminology and notation. A *Cremona transformation* of \mathbb{P}^n is a birational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Its *base locus* $\text{Bs}(f)$ is the subscheme where f is undefined.

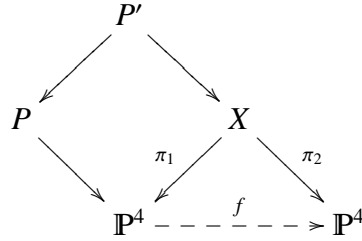
Throughout this paper, we consider the Cremona transformation $f : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ with base locus resolved by blowing up an irreducible surface S , with singular locus consisting of double points. Here a *double point* means a point where the surface has two smooth branches meeting transversally.

Suppose S has δ double points which form a subset $\Delta \subset \mathbb{P}^4$. The blowup of \mathbb{P}^4 along S can be factored as follows:

- (1) Blow up \mathbb{P}^4 along Δ , introducing δ exceptional divisors E_1, \dots, E_δ isomorphic to \mathbb{P}^3 . Let P denote the resulting fourfold and S' the proper transform of S , which is now smooth.
- (2) Blow up P along S' to obtain P' . Let E denote the resulting exceptional divisor and E'_1, \dots, E'_δ the proper transforms of the first group of exceptional divisors. Each E'_i is isomorphic to $E_i \simeq \mathbb{P}^3$ blown up along two skew lines $Q'_i, Q''_i \subset E_i$.
- (3) Each E'_i is a \mathbb{P}^1 -bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed, through each $p \in E_i$ not on Q'_i and Q''_i passes a unique line l intersecting Q'_i and Q''_i . The bundle map is given by $p \mapsto (l \cap Q'_i, l \cap Q''_i) \in Q'_i \times Q''_i$. Blow down each E'_i to $\mathbb{P}^1 \times \mathbb{P}^1$. The resulting X is isomorphic to $\text{Bl}_S \mathbb{P}^4$.

Remark. The blowup $X \rightarrow \mathbb{P}^4$ has a quadric surface Q_i , $i = 1, \dots, \delta$, over each double point of S . Then P' is obtained as the blowup of X along these quadrics.

Let $\pi_1 : X \rightarrow \mathbb{P}^4$ be the blowup along S and $\pi_2 : X \rightarrow \mathbb{P}^4$ the resolution of f so that $\pi_2 = \pi_1 \circ f$. We organize these maps into a diagram:



Note that, by the definition of blowup, X is exactly the graph of f . Let L (resp. M) denote the divisor of the hyperplane class of the left (resp. right) \mathbb{P}^4 . We also use L (resp. M) to denote its pullbacks to X , P and P' (resp. X and P').

It's clear that $L^4 = 1$. We have

$$(1.1) \quad M^4 = 1$$

on X as f is birational. We define n by

$$(1.2) \quad L^3 M = n$$

and ξ by

$$(1.3) \quad LM^3 = \xi.$$

We may interpret n and ξ as the degree of the homogeneous forms inducing f and f^{-1} respectively. Define m as the multiplicity of S in the base locus. It's clear that $M = nL - m(E + 2\Sigma_{i=1}^{\delta} E'_i)$ on P' . Since a nondegenerate subvariety in projective space has degree greater than one, the linear system in $|M|$ inducing $P' \rightarrow \mathbb{P}^4$ must be complete. Thus we have

$$(1.4) \quad h^0(P', M) = 5.$$

We use these equations in our classification of Cremona transformations below.

1.2. Computing the intersection numbers. Let Σ denote the normalization of S and let K_{Σ} be the canonical class. Then the blowup of Σ along the preimage of the double points is isomorphic to S' . Because $S' \rightarrow \Sigma \rightarrow S$ is birational, we consider a general sectional curve $C \subset S$ also curves on Σ and S' . Define $d = C^2 = \deg S$. Note that $E_i \cap S' = Q'_i \cup Q''_i$ are exactly the exceptional curves on S' over the i -th double point.

Lemma 1.1. *We have $LE'_i = 0$. We also have $E^3E'_i = -4$, $E^2E'^2_i = 2$, $EE'^3_i = 0$ and $E'^4_i = -1$.*

Proof. First, $LE'_i = 0$ since their intersection is empty.

Recall that E'_i is isomorphic to $E_i \simeq \mathbb{P}^3$ blown up at skew lines Q'_i and Q''_i . Write $\text{Pic}(E'_i) = \langle H, \tilde{Q}', \tilde{Q}'' \rangle$ where H is the polarization from \mathbb{P}^3 while \tilde{Q}' and \tilde{Q}'' are the exceptional divisors over the lines. We clearly have $\tilde{Q}'\tilde{Q}'' = 0$ and $\tilde{Q}'H = \tilde{Q}''H = 0$. Since $N_{Q'_i/\mathbb{P}^3} = \mathcal{O}_{Q'_i}(1) \oplus \mathcal{O}_{Q'_i}(1)$ then writing $\zeta = c_1(\mathcal{O}_{\mathbb{P}(N_{Q'_i/\mathbb{P}^3})}(1))$ we obtain $\zeta^2 + 2H\zeta = 0$ in the Chow group of $\tilde{Q}' = \mathbb{P}(N_{Q'_i/\mathbb{P}^3})$. We have $\tilde{Q}'|_{\tilde{Q}'} = -\zeta$ so that

$$\tilde{Q}'^2H = -\zeta H = -1, \quad \tilde{Q}'^3 = \zeta^2 = -2H\zeta = -2.$$

We have $N_{E'_i/P'} = \mathcal{O}(-H)$ and $E|_{E'_i} = \tilde{Q}' + \tilde{Q}''$. Thus we obtain

$$\begin{aligned} E^3E'_i &= (\tilde{Q}' + \tilde{Q}'')^3 = \tilde{Q}'^3 + \tilde{Q}''^3 = -4 \\ E^2E'^2_i &= (\tilde{Q}' + \tilde{Q}'')^2(-H) = 2 \\ EE'^3_i &= (\tilde{Q}' + \tilde{Q}'')(-H)^2 = 0 \\ E'^4_i &= (-H)^3 = -1. \end{aligned}$$

□

Lemma 1.2. *The intersection numbers between L and E are*

- (1) $L^3E = 0$ and $L^2E^2 = -d$
- (2) $LE^3 = -5d - K_\Sigma C$
- (3) $E^4 = -15d - 5K_\Sigma C - c_2(\Sigma) + 6\delta$
- (3') $E^4 = d^2 - 25d - 10K_\Sigma C - K_\Sigma^2 + 4\delta$

Proof. $L^3E = 0$ since a general line doesn't intersect S . We have $L^2E^2 = -\deg S' = -d$.

Assume that $C = S \cap L$ for some hyperplane $L \simeq \mathbb{P}^3$. Then $LE^3 = s(C, L)_0$ the zeroth Segre class of C in L , which equals $[c(N_{C/L})^{-1}]_0 = [c(C)c(\iota^*T_{\mathbb{P}^3})]_0 = [(C) - K_\Sigma C - C^2][(C) - 4d]_0 = -5d - K_\Sigma C$.

We have $E^4 = -s(S', P)_0 = -[c(N_{S'/P})^{-1}]_0 = -[c(S')c(P)|_{S'}^{-1}]_0$. Let $\epsilon : P \rightarrow \mathbb{P}^4$ be the blowup. By the blowup formula for Chern classes

$$\begin{aligned} c(P) &= \epsilon^*c(\mathbb{P}^4) + (1 + \Sigma_i E_i)(1 - \Sigma_i E_i)^4 - 1 \\ &= ([P] + L)^5 + \Sigma_i(-3E_i + 2E_i^2 - 4E_i^3 + 3E_i^4). \end{aligned}$$

Thus

$$\begin{aligned} c(P)|_{S'} &= ([S'] + 5C + 10C^2) + \Sigma_i[-3(Q'_i + Q''_i) + 2(Q'^2_i + Q''^2_i)] \\ &= [S'] + 5C - 3\Sigma_i(Q'_i + Q''_i) + 10d - 4\delta. \end{aligned}$$

And thus

$$c(P)|_{S'}^{-1} = [S'] - 5C + 3\Sigma_i(Q'_i + Q''_i) + 15d - 14\delta.$$

Let $\tau : S' \rightarrow \Sigma$ be the blowup. Then

$$c(S') = [S'] - \tau^* K_\Sigma - \Sigma_i(Q'_i + Q''_i) + c_2(\Sigma) + 2\delta.$$

Multiply the results to get $E^4 = -15d - 5K_\Sigma C - c_2(\Sigma) + 6\delta$.

Another expression for E^4 is derived from

$$-[c(N_{S'/P})^{-1}]_0 = c_2(N_{S'/P}) - c_1(N_{S'/P})^2.$$

We have $c_2(N_{S'/P}) = d^2 - 4\delta$. On the other hand

$$\begin{aligned} c_1(N_{S'/P}) &= c_1(T_P)|_{S'} - c_1(T_{S'}) \\ &= -(-5L + 3\Sigma_i E_i)|_{S'} - (-K_{S'}) \\ &= 5C + \tau^* K_\Sigma - 2\Sigma_i(Q'_i + Q''_i). \end{aligned}$$

Thus

$$c_1(N_{S'/P})^2 = 25d + 10K_\Sigma C + K_\Sigma^2 - 8\delta.$$

Therefore

$$\begin{aligned} E^4 &= (d^2 - 4\delta) - (25d + 10K_\Sigma C + K_\Sigma^2 - 8\delta) \\ &= d^2 - 25d - 10K_\Sigma C - K_\Sigma^2 + 4\delta. \end{aligned}$$

□

2. CONSTRUCTION OF OUR EXAMPLE

In this section, we use Mukai's construction [Muk88] to build up an explicit example of a degree 12 K3 surface $R \subset \mathbb{P}^7$ together with three points $p_1, p_2, p_3 \in R$. This example helps us prove the following theorem:

Theorem 2.1. *Let $R \subset \mathbb{P}^7$ be a generic K3 surface of degree 12 and $\Pi := \{p_1, p_2, p_3\} \subset R$ a generic triple of points.*

- (1) *projection from Π maps R to a surface $S \subset \mathbb{P}^4$ with three double points;*
- (2) *the complete linear system M of quartics vanishing along S cut out S scheme-theoretically;*
- (3) *M induces a birational map $f : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$.*
- (4) *The base locus of the inverse f^{-1} is also a projection of a degree 12 K3 surface from three points.*

We prove Theorem 2.1 in the end of section.

2.1. Orthogonal Grassmannian. Let V be a 10-dimensional vector space equipped with a nondegenerate quadratic form q . The 5-subspaces of V isotropic with respect to q form a subvariety \mathcal{S} of the Grassmannian $G(5, V)$. It has two components \mathcal{S}^+ and \mathcal{S}^- which are isomorphic to each other. They are called *orthogonal Grassmannians* and are denoted by $OG(5, V)$.

Fix a 5-subspace $W \in \text{OG}(5, V)$ and let W^* be its orthogonal complement with respect to q . Then $\text{OG}(5, V)$ can be identified scheme theoretically as the zero locus in

$$\mathbb{P}(\mathbb{C} \oplus \bigwedge^2 W \oplus \bigwedge^4 W) \simeq \mathbb{P}^{15}$$

of the quadratic form [Kap13, 3.2]

$$(2.1) \quad \begin{array}{ccc} \det W \oplus \bigwedge^2 W \oplus W^* & \longrightarrow & \bigwedge^4 W \oplus W \\ (x, \Omega, v) & \longmapsto & (x(v) + \frac{1}{2}\Omega \wedge \Omega, \Omega(v)). \end{array}$$

Here we choose an isomorphism $\mathbb{C} \simeq \det W$. This induces an isomorphism $\bigwedge^4 W \simeq W^*$.

Let $\mathbf{x} = (x_0, \dots, x_{15})$ be the homogeneous coordinate for \mathbb{P}^{15} . Then (2.1) can be explicitly written down as ten quadrics:

$$(2.2) \quad \begin{array}{l} x_0x_{11} + x_5x_{10} - x_6x_9 + x_7x_8 \\ x_0x_{12} + x_2x_{10} - x_3x_9 + x_4x_8 \\ x_0x_{13} + x_1x_{10} - x_3x_7 + x_4x_6 \\ x_0x_{14} + x_1x_9 - x_2x_7 + x_4x_5 \\ x_0x_{15} + x_1x_8 - x_2x_6 + x_3x_5 \end{array} \left| \begin{array}{l} -x_1x_{12} + x_2x_{13} - x_3x_{14} + x_4x_{15} \\ x_1x_{11} - x_5x_{13} + x_6x_{14} - x_7x_{15} \\ -x_2x_{11} + x_5x_{12} - x_8x_{14} + x_9x_{15} \\ x_3x_{11} - x_6x_{12} + x_8x_{13} - x_{10}x_{15} \\ -x_4x_{11} + x_7x_{12} - x_9x_{13} + x_{10}x_{14}. \end{array} \right.$$

2.2. An explicit example. Mukai [Muk88, §3] proves that a generic K3 surface of degree 12 appears as a linear section of $\text{OG}(5, V)$ and vice versa.

For example, the $\mathbb{P}^7 \subset \mathbb{P}^{15}$ spanned by the rows of the 8×16 matrix

$$\mathbf{H} = \begin{pmatrix} -1 & 3 & 2 & 0 & 2 & -3 & -1 & 0 & 3 & 1 & 0 & 3 & 0 & 2 & 0 & -3 \\ 1 & 0 & -3 & 0 & -2 & 1 & 1 & 0 & -2 & -1 & -1 & -1 & 0 & 4 & 0 & 2 \\ -1 & -3 & -2 & 0 & -3 & 0 & 3 & 2 & -1 & -3 & -1 & 2 & -1 & 2 & 0 & 3 \\ 3 & 0 & 0 & 2 & 2 & 3 & 0 & 1 & 2 & -1 & 0 & 2 & -1 & -2 & 2 & 3 \\ 0 & -1 & 1 & -1 & 0 & 1 & -3 & 3 & 2 & 2 & 1 & 3 & 0 & -3 & 0 & -3 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

cut out a degree 12 K3 surface R on $\text{OG}(5, V)$. More explicitly, let $\mathbf{z} = (z_0, \dots, z_7)$ be the homogeneous coordinate for \mathbb{P}^7 . We define the inclusion $\iota : \mathbb{P}^7 \hookrightarrow \mathbb{P}^{15}$ by

$$\mathbf{x} = \mathbf{z} \cdot \mathbf{H}.$$

Then we get $R = \iota^{-1}(\text{OG}(5, V))$.

The last three rows of \mathbf{H} are chosen as solutions of (2.2) so that they form a triple of points $\Pi = \{p_1, p_2, p_3\} \subset R$. With this choice the projection from Π is exactly the map

$$\pi : \begin{array}{ccc} \mathbb{P}^7 & \dashrightarrow & \mathbb{P}^4 \\ (z_0, \dots, z_7) & \mapsto & (z_0, \dots, z_4) \end{array}$$

which takes R to $S = \pi(R)$.

We manipulate this example in a computer algebra system¹ over the finite field \mathbb{F}_7 of order 7. We compute that S is singular along three double points $\{a_1, a_2, a_3\}$ and is the base locus of a Cremona transformation

$$\bar{f} : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4.$$

Moreover, the base locus of the inverse $(\bar{f})^{-1}$ is again a surface \bar{T} singular along three double points $\{b_1, b_2, b_3\}$. The matrix \mathbf{H} is chosen such that the preimage of $\{a_1, a_2, a_3\}$ on R and the preimage of $\{b_1, b_2, b_3\}$ on the normalization of \bar{T} are \mathbb{F}_7 -rational points. Seven is the lowest characteristic where our computer can find such \mathbf{H} within a reasonable period of time.

2.3. Proof of Theorem 2.1. We prove Theorem 2.1 for our example first.

We confirm the following properties by computer over \mathbb{F}_7 :

- (1) S is singular along three points. The preimage of each singular point on R has two points outside Π . So they are double points.
- (2) The ideal of S is generated by five quartics $\bar{f}_0, \dots, \bar{f}_4$.

Property (1) implies Theorem 2.1(1). Indeed, the double-point formula implies that S is singular along three points counted with multiplicity if its singular locus is 0-dimensional. This implies that the three double points in (1) actually exist over characteristic zero.

The five quartics $\bar{f}_0, \dots, \bar{f}_4$ lift to a basis f_0, \dots, f_4 for the ideal of S over characteristic zero. In particular Theorem 2.1(2) holds. f_0, \dots, f_4 define a rational map

$$f := (f_0, \dots, f_4) : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$$

which reduces to

$$\bar{f} := (\bar{f}_0, \dots, \bar{f}_4) : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$$

over \mathbb{F}_7 . We have $\deg f = M^4$. So f is birational if and only if $M^4 = 1$, and if and only if the equivalent formula (5.2) is satisfied. One can verify it with the help of Lemma 5.5 by inserting our current data: $(n, m, d, \delta) = (4, 1, 9, 3)$. Thus Theorem 2.1(3) holds.

The inverse $(\bar{f})^{-1}$ can be calculated by computer. It consists of five quartics also and the base locus is a surface \bar{T} singular along three points. These are double points since each point has two preimage points on the normalization. By the same reasons as above, the base locus of f^{-1} is again a surface cut out by five quartics and singular along three double points. Then Theorem 2.1(4) follows from Theorem 5.1.

Next we prove Theorem 2.1 for general cases.

It's clear that (1), (2) and (3) of the theorem are open conditions, so they hold for a generic example. As a consequence of Theorem 5.1, property (4) holds once $\text{Bs}(f^{-1})$ is a surface cut out by five quartics and singular along

¹The main program we use in this work is SINGULAR [DGPS15]

three double points. These are open conditions again so Theorem 2.1 holds for a generic example.

3. DERIVED EQUIVALENCES OF K3 SURFACES

Let $f : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ be a Cremona transformation in Theorem 2.1. It has a resolution

$$\begin{array}{ccc} & X & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}^4 & \xrightarrow{f} & \mathbb{P}^4 \end{array}$$

where π_1 is the blowup along $S_L = \text{Bs}(f)$. Let R_L be the K3 surface over S_L . By the same theorem there's another K3 surface R_M over $S_M = \text{Bs}(f^{-1})$. Recall that X is the same as the graph of f , so π_2 is the blowup along S_M .

Theorem 3.1. *The two K3 surfaces R_L and R_M are derived equivalent. They are non-isomorphic if they have Picard number one.*

Corollary 3.2. *There is a birational map $\sigma : R_L^{[3]} \dashrightarrow R_M^{[3]}$ between the Hilbert schemes of length three subschemes.*

3.1. Derived equivalence of K3 surfaces and strategy. Let R and \hat{R} denote K3 surfaces and $T(R)$ and $T(\hat{R})$ the corresponding transcendental lattices. Recall that R and \hat{R} are derived equivalent if and only if $T(R)$ and $T(\hat{R})$ are Hodge isometric [Orl97]. Suppose that R has Picard rank one and has degree $2n$ and $\tau(n)$ is the number of prime factors of n . Then the number of isomorphism classes of K3 surfaces derived equivalent to R is equal to $2^{\tau(n)-1}$ [HLOY03]. Thus a degree 12 K3 surface admits a unique such partner.

Our general approach is to prove that $T(R_L)$ is isometric to $T(R_M)$ by showing that both of them can be identified as the transcendental sublattice of $H^4(X, \mathbb{Z})$. Then we show that the induced isomorphism on the discriminant groups is nontrivial, which implies that R_L and R_M are not isomorphic to each other.

3.2. The middle cohomology of X . Retain the notation of Section 1. Let H_L be the polarization of R_L . Let F_1, F_2 and F_3 be the exceptional curves from the projection $R_L \dashrightarrow S_L$. We consider H_L, F_1, F_2 and F_3 as curves on S_L . Their strict transforms $\tilde{H}_L, \tilde{F}_1, \tilde{F}_2, \tilde{F}_3$ on X together with L^2 and the quadrics Q_1, Q_2, Q_3 form a rank 8 sublattice $A_L(X) \subset H^4(X, \mathbb{Z})$. We have

$$\tilde{H}_L^2 = -H_L^2 = -12, \quad \tilde{F}_i^2 = -F_i^2 = 1 \quad \text{and} \quad Q_i^2 = -E_i'^4 = 1$$

where $i = 1, 2, 3$. These classes are mutually disjoint, so the intersection matrix for $A_L(X)$ is

$$(3.1) \quad \begin{array}{c|cccc} & L^2 & \widetilde{H}_L & \widetilde{F}_{1,2,3} & \widetilde{Q}_{1,2,3} \\ \hline L^2 & 1 & & & \\ \widetilde{H}_L & & -12 & & \\ \widetilde{F}_{1,2,3} & & & I_{3 \times 3} & \\ \widetilde{Q}_{1,2,3} & & & & I_{3 \times 3} \end{array}$$

where $I_{3 \times 3}$ is the identity matrix of rank 3.

Lemma 3.3. *There is a decomposition*

$$H^4(X, \mathbb{Z}) \simeq H^4(X, \mathbb{Z})_{\text{alg}} \oplus_{\perp} T(R_L)(-1).$$

where $H^4(X, \mathbb{Z})_{\text{alg}}$ is the sublattice spanned by algebraic classes. We have

$$H^4(X, \mathbb{Z})_{\text{alg}} = A_L(X)$$

when R_L has Picard number one. Here we use $\Lambda(-1)$ to denote a lattice Λ equipped with the negative of its original product.

Proof. We apply the blowup formula for cohomology to the composition $P' \rightarrow P \rightarrow \mathbb{P}^4$ and the map $P' \rightarrow X$ to obtain two decompositions for $H^4(P', \mathbb{Z})$. Then we compare them to get our result.

Let $S'_L \subset P$ be the strict transform of S_L . Recall that S'_L is isomorphic to R_L blown up at $3+6=9$ points, where 3 are from the projection $R_L \dashrightarrow S_L$ while 6 are from the resolution $S'_L \rightarrow S_L$. Then

$$H^2(S'_L, \mathbb{Z}) \simeq \langle F_i, Q'_i, Q''_i \rangle_{i=1,2,3} \oplus H^2(R_L, \mathbb{Z}).$$

Let \widetilde{Q}'_i and \widetilde{Q}''_i be the strict transform of Q'_i and Q''_i on P' . Since $P' \rightarrow P$ is the blowup along S'_L , we have

$$(3.2) \quad \begin{aligned} H^4(P', \mathbb{Z}) &\simeq H^4(P, \mathbb{Z}) \oplus H^2(S'_L, \mathbb{Z})(-1) \\ &\simeq \langle L^2, E_i'^2, \widetilde{F}_i, \widetilde{Q}'_i, \widetilde{Q}''_i \rangle_{i=1,2,3} \oplus H^2(R_L, \mathbb{Z})(-1). \end{aligned}$$

For every i , we have

$$\widetilde{Q}_i'^2 = -Q_i'^2 = 1, \quad \widetilde{Q}_i''^2 = -Q_i''^2 = 1, \quad E_i'^2 \widetilde{Q}_i' = E_i'^2 \widetilde{Q}_i'' = 0$$

and $E_i'^4 = -1$. With these it's straightforward to prove the isometry

$$\langle E_i'^2, \widetilde{Q}_i', \widetilde{Q}_i'' \rangle \simeq \langle E_i'^2 + \widetilde{Q}_i', E_i'^2 + \widetilde{Q}_i'', E_i'^2 + \widetilde{Q}_i' + \widetilde{Q}_i'' \rangle,$$

whence (3.2) equals

$$(3.3) \quad H^4(P', \mathbb{Z}) \simeq \langle E_i'^2 + \widetilde{Q}_i', E_i'^2 + \widetilde{Q}_i'', E_i'^2 + \widetilde{Q}_i' + \widetilde{Q}_i'', L^2, \widetilde{F}_i \rangle_{i=1,2,3} \oplus H^2(R_L, \mathbb{Z})(-1).$$

By the description of the map $E_i \dashrightarrow Q_i$, the two fiber classes on $Q_i \simeq \mathbb{P}^1 \times \mathbb{P}^1$ pullback to hyperplanes in E_i containing either Q'_i or Q''_i , which

correspond to the classes $-E_i'^2 - \widetilde{Q}_i'$ or $-E_i'^2 - \widetilde{Q}_i''$ on P' , respectively. The map $P' \rightarrow X$ is the blowup along Q_i , $i = 1, 2, 3$, so

$$(3.4) \quad H^4(P', \mathbb{Z}) \simeq \left\langle E_i'^2 + \widetilde{Q}_i', E_i'^2 + \widetilde{Q}_i'' \right\rangle_{i=1,2,3} \oplus H^4(X, \mathbb{Z})$$

Combining (3.3) and (3.4) we get

$$\begin{aligned} H^4(X, \mathbb{Z}) &\simeq \left\langle E_i'^2 + \widetilde{Q}_i', E_i'^2 + \widetilde{Q}_i'' \right\rangle_{i=1,2,3}^\perp \\ &\simeq \left\langle E_i'^2 + \widetilde{Q}_i' + \widetilde{Q}_i'', L^2, \widetilde{F}_i \right\rangle_{i=1,2,3} \oplus H^2(R_L, \mathbb{Z})(-1). \end{aligned}$$

Both Q_i and $E_i'^2 + \widetilde{Q}_i' + \widetilde{Q}_i''$ are orthogonal to L^2 , $\widetilde{F}_{i=1,2,3}$ and $H^2(R_L, \mathbb{Z})$, and $Q_i^2 = (E_i'^2 + \widetilde{Q}_i' + \widetilde{Q}_i'')^2 = 1$, so $Q_i = \pm(E_i'^2 + \widetilde{Q}_i' + \widetilde{Q}_i'')$. Therefore

$$\begin{aligned} H^4(X, \mathbb{Z}) &\simeq \left\langle Q_i, L^2, \widetilde{F}_i \right\rangle_{i=1,2,3} \oplus H^2(R_L, \mathbb{Z})(-1) \\ &\simeq \left\langle Q_i, L^2, \widetilde{F}_i \right\rangle_{i=1,2,3} \oplus NS(R_L)(-1) \oplus_\perp T(R_L)(-1) \\ &\simeq H^4(X, \mathbb{Z})_{alg} \oplus_\perp T(R_L) \end{aligned}$$

where $NS(R_L)$ is the Néron-Severi lattice of R_L .

When R_L has Picard number one, we have $NS(R_L)(-1) \simeq \langle \widetilde{H}_L \rangle$. In this case

$$H^4(X, \mathbb{Z})_{alg} \simeq \left\langle Q_i, L^2, \widetilde{F}_i, \widetilde{H}_L \right\rangle_{i=1,2,3} = A_L(X).$$

□

Lemma 3.3 also proves the decomposition

$$H^4(X, \mathbb{Z}) \simeq H^4(X, \mathbb{Z})_{alg} \oplus_\perp T(R_M)(-1)$$

from the side of f^{-1} . So there is an isometry

$$T(R_L) \simeq H^4(X, \mathbb{Z})_{alg}^\perp(-1) \simeq T(R_M)$$

which allows us to conclude that

Proposition 3.4. *R_L and R_M are derived equivalent.*

3.3. The discriminant groups. For an arbitrary lattice Λ with dual lattice $\Lambda^* := \text{Hom}(\Lambda, \mathbb{Z})$, we denote by $d\Lambda := \Lambda^*/\Lambda$ its discriminant group.

Let $A_M(X)$ be the lattice constructed in the same way as $A_L(X)$ from the side of f^{-1} . Assume R_L and R_M have Picard number one. Then Lemma 3.3 implies that there is an isometry

$$\varphi : A_M(X) \oplus_\perp T(R_M)(-1) \xrightarrow{\sim} A_L(X) \oplus_\perp T(R_L)(-1)$$

such that $\varphi = \varphi_A \oplus \varphi_T$ with respect to the decompositions. It induces the commutative diagram

$$\begin{array}{ccc} dA_M(X) & \xrightarrow[\sim]{\varphi_{A*}} & dA_L(X) \\ \sim \downarrow & & \sim \downarrow \\ dT(R_M) & \xrightarrow[\sim]{\varphi_{T*}} & dT(R_L). \end{array}$$

These groups are all isomorphic to $\mathbb{Z}/12\mathbb{Z}$. From the intersection matrix (3.1) we know that $dA_L(X)$ is generated by $-\widetilde{H}_L/12$. Similarly, $dA_M(X)$ is generated by $-\widetilde{H}_M/12$ where H_M is the polarization of R_M and \widetilde{H}_M is the strict transform on X .

Lemma 3.5. *We have the following equations in $H^4(X, \mathbb{Z})$*

- (1) $M^2 = 7L^2 - 3\widetilde{H}_L + 4(\widetilde{F}_1 + \widetilde{F}_2 + \widetilde{F}_3) + 2(Q_1 + Q_2 + Q_3)$
- (2) $\widetilde{H}_M = 36L^2 - 17\widetilde{H}_L + 24(\widetilde{F}_1 + \widetilde{F}_2 + \widetilde{F}_3) + 12(Q_1 + Q_2 + Q_3)$

Proof. Assume that

$$M^2 = aL^2 + b\widetilde{H}_L + f_1\widetilde{F}_1 + f_2\widetilde{F}_2 + f_3\widetilde{F}_3 + g_1Q_1 + g_2Q_2 + g_3Q_3.$$

Then $a = L^2M^2 = 7$. For $i = 1, 2, 3$, we have

$$\begin{aligned} g_i &= M^2Q_i = -M^2E_i'^2 \\ &= -(4L - E - 2\Sigma_j E_j')^2 E_i'^2 = -(-E - 2\Sigma_j E_j')^2 E_i'^2 \\ &= -E^2 E_i'^2 - 4EE_i'^3 - 4E_i'^4 = -2 - 0 + 4 = 2. \end{aligned}$$

Let \widetilde{C} be the strict transform of the sectional curve C on P' . Note that

$$LM = 4L^2 - LE = 4L^2 - \widetilde{C} = 4L^2 - \widetilde{H}_L + \Sigma_i \widetilde{F}_i.$$

So

$$\begin{aligned} 4 &= LM^3 = (LM)M^2 \\ &= (4L^2 - \widetilde{H}_L + \Sigma_i \widetilde{F}_i)(7L^2 + b\widetilde{H}_L + \Sigma_i f_j \widetilde{F}_j + 2\Sigma_k Q_k) \\ &= 28 + 12b + f_1 + f_2 + f_3. \end{aligned}$$

Thus

$$(3.5) \quad f_1 + f_2 + f_3 = -12b - 24.$$

We also have

$$\begin{aligned} 1 &= M^4 = (7L^2 + b\widetilde{H}_L + \Sigma_i f_j \widetilde{F}_j + 2\Sigma_k Q_k)^2 \\ &= 49 - 12b^2 + f_1^2 + f_2^2 + f_3^2 + 12 \end{aligned}$$

which is equivalent to

$$(3.6) \quad f_1^2 + f_2^2 + f_3^2 = 12b^2 - 60.$$

By the Cauchy-Schwarz inequality

$$(3.7) \quad \begin{aligned} (f_1 + f_2 + f_3)^2 &= ((1, 1, 1) \cdot (f_1, f_2, f_3))^2 \\ &\leq (1, 1, 1)^2 (f_1, f_2, f_3)^2 = 3(f_1^2 + f_2^2 + f_3^2). \end{aligned}$$

Apply (3.5) and (3.6) we get $(-12b - 24)^2 \leq 3(12b^2 - 60)$, i.e.

$$3b^2 + 16b + 21 = (3b + 7)(b + 3) \leq 0.$$

The only integer solution is $b = -3$. Because (3.7) becomes an equality in this case, we have $(f_1, f_2, f_3) = f(1, 1, 1)$ for some integer f . We obtain $f = 4$ by setting $b = -3$ in (3.5). As a result,

$$M^2 = 7L^2 - 3\widetilde{H}_L + 4(\widetilde{F}_1 + \widetilde{F}_2 + \widetilde{F}_3) + 2(Q_1 + Q_2 + Q_3).$$

Next, assume that

$$\widetilde{H}_M = aL^2 + b\widetilde{H}_L + f_1\widetilde{F}_1 + f_2\widetilde{F}_2 + f_3\widetilde{F}_3 + g_1Q_1 + g_2Q_2 + g_3Q_3.$$

By symmetry, $\widetilde{H}_L L^n M^{2-n} = \widetilde{H}_M M^n L^{2-n}$ for $n = 0, 1, 2$. In particular,

$$a = \widetilde{H}_M L^2 = \widetilde{H}_L M^2 = -3\widetilde{H}_L^2 = 36.$$

We have

$$\begin{aligned} 12 &= \widetilde{H}_L(4L^2 - \widetilde{H}_L + \Sigma_i \widetilde{F}_i) = \widetilde{H}_L(LM) = \widetilde{H}_M(ML) \\ &= (36L^2 + b\widetilde{H}_L + \Sigma_i f_i \widetilde{F}_i + \Sigma_j g_j Q_j)(4L^2 - \widetilde{H}_L + \Sigma_i \widetilde{F}_i) \\ &= 144 + 12b + f_1 + f_2 + f_3. \end{aligned}$$

Rearrange to obtain

$$(3.8) \quad f_1 + f_2 + f_3 = -12b - 132.$$

Applying the symmetry again, we get

$$\begin{aligned} 0 &= \widetilde{H}_L L^2 = \widetilde{H}_M M^2 \\ &= (36L^2 + b\widetilde{H}_L + \Sigma_i f_i \widetilde{F}_i + \Sigma_j g_j Q_j)(7L^2 - 3\widetilde{H}_L + 4\Sigma_i \widetilde{F}_i + 2\Sigma_j Q_j) \\ &= 252 + 36b + 4(f_1 + f_2 + f_3) + 2(g_1 + g_2 + g_3) \end{aligned}$$

whence

$$2(f_1 + f_2 + f_3) + (g_1 + g_2 + g_3) = -18b - 126$$

and combining with (3.8) gives

$$(3.9) \quad g_1 + g_2 + g_3 = 6b + 138.$$

We also have

$$\begin{aligned} -12 &= \widetilde{H}_L^2 = \widetilde{H}_M^2 = (36L^2 + b\widetilde{H}_L + \Sigma_i f_i \widetilde{F}_i + \Sigma_j g_j Q_j)^2 \\ &= 1296 - 12b^2 + f_1^2 + f_2^2 + f_3^2 + g_1^2 + g_2^2 + g_3^2, \end{aligned}$$

from which we obtain

$$(3.10) \quad f_1^2 + f_2^2 + f_3^2 + g_1^2 + g_2^2 + g_3^2 = 12b^2 - 1308.$$

By the Cauchy-Schwarz inequality,

$$(3.11) \quad \begin{aligned} (-12b - 132)^2 &= (f_1 + f_2 + f_3)^2 = ((1, 1, 1) \cdot (f_1, f_2, f_3))^2 \\ &\leq (1, 1, 1)^2 (f_1, f_2, f_3)^2 = 3(f_1^2 + f_2^2 + f_3^2) \end{aligned}$$

and

$$(3.12) \quad \begin{aligned} (6b + 138)^2 &= (g_1 + g_2 + g_3)^2 = ((1, 1, 1) \cdot (g_1, g_2, g_3))^2 \\ &\leq (1, 1, 1)^2 (g_1, g_2, g_3)^2 = 3(g_1^2 + g_2^2 + g_3^2). \end{aligned}$$

Add the two inequalities and then apply (3.10) to get

$$(3.13) \quad \begin{aligned} (-12b - 132)^2 + (6b + 138)^2 \\ \leq 3(f_1^2 + f_2^2 + f_3^2 + g_1^2 + g_2^2 + g_3^2) = 3(12b^2 - 1308) \end{aligned}$$

which can be arranged as

$$2b^2 + 67b + 561 = (2b + 33)(b + 17) \leq 0.$$

The only integer solution is $b = -17$ which makes (3.13) an equality. This forces (3.11) and (3.12) to be equalities also. Therefore $(f_1, f_2, f_3) = f(1, 1, 1)$ and $(g_1, g_2, g_3) = g(1, 1, 1)$ for some integers f and g . We get $f = 24$ from (3.8) and $g = 12$ from (3.9). As a consequence,

$$\widetilde{H}_M = 36L^2 - 17\widetilde{H}_L + 24(\widetilde{F}_1 + \widetilde{F}_2 + \widetilde{F}_3) + 12(Q_1 + Q_2 + Q_3).$$

□

Proposition 3.6. *The isomorphism $\varphi_{A*} : dA_M(X) \xrightarrow{\sim} dA_L(X)$ equals multiplication by 7 on $\mathbb{Z}/12\mathbb{Z}$.*

Proof. Recall that φ_A acts as the identity map on $H^4(X, \mathbb{Z})_{alg}$. So $\varphi_A(\widetilde{H}_M) = \widetilde{H}_M$. By Lemma 3.5 we have

$$\varphi_A(\widetilde{H}_M) = 36L^2 - 17\widetilde{H}_L + 24(\widetilde{F}_1 + \widetilde{F}_2 + \widetilde{F}_3) + 12(Q_1 + Q_2 + Q_3).$$

as a map from $A_M(X)$ to $A_L(X)$. Therefore

$$\begin{aligned} \varphi_{A*}(-\tfrac{1}{12}\widetilde{H}_M) &= -3L^2 + \tfrac{17}{12}\widetilde{H}_L - 2(\widetilde{F}_1 + \widetilde{F}_2 + \widetilde{F}_3) - (Q_1 + Q_2 + Q_3) \\ &= -17 \cdot (-\tfrac{1}{12}\widetilde{H}_L) \pmod{A_L(X)} \\ &= 7 \cdot (-\tfrac{1}{12}\widetilde{H}_L) \pmod{A_L(X)}. \end{aligned}$$

□

3.4. Proofs of Theorem 3.1 and its Corollary. We first prove the theorem.

The derived equivalence follows from Proposition 3.4. Note that this implies that the Picard numbers of R_L and R_M are the same.

Assume R_L and R_M have Picard number one. Suppose they are isomorphic. Then there is an isometry

$$\theta : T(R_L) \xrightarrow{\sim} T(R_M)$$

which induces the isomorphism

$$\begin{aligned} \theta_* : dT(R_L) &\xrightarrow{\sim} dT(R_M) \\ -\tfrac{\widetilde{H}_L}{12} &\mapsto -\tfrac{\widetilde{H}_M}{12} \end{aligned}$$

under the identifications $dT(R_L) \simeq dA_L(X)$ and $dT(R_M) \simeq dA_M(X)$.

By Proposition 3.6, the composition $\varphi_A \circ \theta$ is an automorphism on $T(R_L)$ acting as multiplication by 7 on $dT(R_L)$. This contradicts the fact that the only automorphism on $T(R_L)$ is the identity [Ogu02]. Hence R_L and R_M can't be isomorphic to each other.

Next we prove the corollary.

The corollary is trivial if R_L and R_M are isomorphic, so we assume that they are non-isomorphic.

Given a generic triple of points $\Pi_L \in R_L^{[3]}$, we determine a degree 12 K3 surface R_M and a triple of points $\Pi_M \in R_M^{[3]}$ through the following steps:

- (1) Project R_L from Π_L to obtain $S_L \subset \mathbb{P}^4$, whose ideal defines a Cremona transformation $f : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$.
- (2) The base locus of f^{-1} is a surface S_M singular along three double points. Normalize S_M to get Σ_M .
- (3) Σ_M is the blowup of a degree 12 K3 surface R_M along three points.

The three exceptional curves on Σ_M are contracted to $\Pi_M \in R_M^{[3]}$.

Recall that a pair of derived equivalent K3 surfaces of degree 12 uniquely determines each other up to isomorphism. So R_M is independent of the choice of $\Pi_L \in R_L^{[3]}$ by Theorem 3.1. Hence there is a rational map

$$\begin{aligned} \sigma : R_L^{[3]} &\dashrightarrow R_M^{[3]} \\ \Pi_L &\mapsto \Pi_M. \end{aligned}$$

It is birational because Π_L is uniquely determined by Π_M through the same process as above.

Remark. Kuznetsov [Kuz06, §6.2] has interpreted this derived equivalence via Homological Projective Duality; it would be interesting to construct the Cremona transformation through this mechanism.

4. ZERO DIVISORS IN THE GROTHENDIECK RING

Let $K_0(\text{Var}/\mathbb{C})$ denote the Grothendieck ring of complex algebraic varieties. It is the abelian group generated by isomorphism classes of complex algebraic varieties subject to the relation

$$[Z] = [U] + [Z - U]$$

where U is an open subvariety of Z . The multiplication is induced by the Cartesian product:

$$[X][Y] = [X \times Y]$$

which is associative, commutative with unit $1 = [\text{Spec } \mathbb{C}]$. More generally, if $Z \rightarrow X$ is a Zariski locally trivial bundle with fibers isomorphic to Y , by stratifying the base it's easy to prove that

$$[X][Y] = [Z].$$

Let $\mathbb{L} = [A^1]$ be the class of the affine line in $K_0(\text{Var}/\mathbb{C})$. Consider a pair of non-isomorphic smooth projective varieties X and Y which are derived equivalent. It is interesting to know if there exists $k \geq 0$ satisfying

$$(4.1) \quad ([X] - [Y])\mathbb{L}^k = 0$$

and what the minimal k is if it exists [KS16].

When X is a generic K3 surface of degree 12, Ito, Miura, Okawa and Ueda [IMOU16b] proves that there exists Y non-trivially derived equivalent to X such that (4.1) holds for $k = 3$. Actually, it can be improved to $k = 1$ straightforwardly from the point of view of the Cremona transformation.

Theorem 4.1. *Let R_L and R_M be a generic pair of K3 surfaces associated with our Cremona transformation. Then we have*

$$([R_L] - [R_M])\mathbb{L} = 0.$$

in $K_0(\text{Var}/\mathbb{C})$. The relation is minimal in the sense that $[R_L] - [R_M] \neq 0$.

Proof. Recall that Σ_L is the normalization of S_L as well as the blowup of R_L at three points. Hence

$$[S_L] = [\Sigma_L] - 3 = [R_L] + 3\mathbb{L} - 3.$$

From the blowup $\pi_1 : X \rightarrow \mathbb{P}^4$ we obtain

$$(4.2) \quad \begin{aligned} [X] &= ([\mathbb{P}^4] - [S_L]) + [\pi_1^{-1}(S_L)] \\ &= ([\mathbb{P}^4] - [S_L]) + (([S_L] - 3)[\mathbb{P}^1] + [Q_1] + [Q_2] + [Q_3]) \\ &= ([\mathbb{P}^4] - [S_L]) + (([S_L] - 3)[\mathbb{P}^1] + 3[\mathbb{P}^1]^2) \\ &= [\mathbb{P}^4] + 3[\mathbb{P}^1]([\mathbb{P}^1] - 1) + [S_L]([\mathbb{P}^1] - 1) \\ &= [\mathbb{P}^4] + 3[\mathbb{P}^1]\mathbb{L} + [S_L]\mathbb{L} \\ &= [\mathbb{P}^4] + 3[\mathbb{P}^1]\mathbb{L} + [R_L]\mathbb{L} + 3\mathbb{L}^2 - 3\mathbb{L}. \end{aligned}$$

By symmetry, we also have

$$(4.3) \quad [X] = [\mathbb{P}^4] + 3[\mathbb{P}^1]\mathbb{L} + [R_M]\mathbb{L} + 3\mathbb{L}^2 - 3\mathbb{L}.$$

Subtracting (4.3) from (4.2) we get

$$([R_L] - [R_M])\mathbb{L} = 0.$$

Next we show that $[R_L] \neq [R_M]$, and it is sufficient to show that $[R_L] \neq [R_M]$ modulo \mathbb{L} . According to [LL03], $[R_L] = [R_M] \pmod{\mathbb{L}}$ if and only if R_L and R_M are stably birational. Because a K3 surface is not rationally connected, this implies that R_L and R_M are birational and thus isomorphic, contradicting to Theorem 3.1. \square

5. EXCLUSION OF ALTERNATIVE CONSTRUCTIONS

This section shows that there exists just one class of Cremona transformations of \mathbb{P}^4 that can be resolved by blowing up an irreducible surface S with double points, i.e., the class constructed in Section 2. Recall that [CK89] classified the case where S is smooth.

Theorem 5.1. *Let $S \subset \mathbb{P}^4$ be an irreducible surface with $\delta > 0$ double points. Assume there exists a Cremona transformation*

$$f : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$$

resolved by blowing up S . Let n and ξ denote the degrees of the homogeneous forms inducing f and f^{-1} respectively, and m the multiplicity of S in the base locus. Then we have

$$n = \xi = 4, \quad m = 1,$$

and S is obtained by projecting a degree 12 K3 surface from three points.

The remainder of this section is devoted to the proof of Theorem 5.1.

5.1. Extracting Diophantine equations. By Lemma 1.1 and 1.2, equation (1.3) can be expressed as

$$(5.1) \quad \xi = n^3 - 3nm^2d + m^3(K_\Sigma C + 5d).$$

Similarly, equation (1.1) can be expressed as

$$(5.2) \quad 1 = n^4 - 6n^2m^2d + 4nm^3(K_\Sigma C + 5d) - m^4(15d + 5K_\Sigma C + c_2(\Sigma) - 6\delta)$$

and equivalently as

$$(5.2') \quad 1 = n^4 - 6n^2m^2d + 4nm^3(K_\Sigma C + 5d) + m^4(d^2 - 25d - 10K_\Sigma C - K_\Sigma^2 + 4\delta).$$

The two formulas follow from the two expressions Lemma 1.2 (3) and (3') for E^4 , respectively. The right hand sides of these equations are arranged as polynomials in n and m . Note that only the coefficients of m^4 reflect the appearance of double points.

5.2. Enumeration of combinatorial cases.

Lemma 5.2. *Only the following (n, m, ξ) can occur.*

	n	m	ξ
(a)	3	1	2
(b)	4	1	4
(c)	7	2	3
(d)	9	2	9
(e)	43	10	7
(f)	24	5	24
(g)	49	10	49

Proof. In the smooth case, the same list [CK89, Theorem 1.6] is obtained by using [CK89, Lemma 0.2] and [CK89, Formulae 0.3]. The proof of the former proceeds unchanged even with the double points. The latter can be derived from (5.1) and (5.2) and only the terms with power of m up to two matter, so double points don't change the result also. Therefore the same elimination process works and we obtain the same list. \square

5.3. Exclusion of cases. Here we show that only Case (b) can occur.

Lemma 5.3. *Cases (c) and (e) do not occur.*

Proof. The proof is similar to the smooth case [CK89, Lemma 3.2].

Assume Case (c) holds. Then (5.1) reduces to

$$2K_{\Sigma}C = 11d - 85.$$

Thus (5.2) reduces to

$$465 = 62d - 2c_2(\Sigma) + 12\delta$$

which is odd on the left and even on the right, a contradiction.

Assume Case (e) holds. Now (5.1) reduces to

$$79d = 795 + 10K_{\Sigma}C,$$

so d is divisible by 5. On the other hand, (5.2) becomes

$$-34188 = -11094d + 1720(K_{\Sigma}C + 5d) - 100(15d + 5K_{\Sigma}C + c_2(\Sigma) - 6\delta).$$

Note that 5 divides the right but not the left, a contradiction. \square

Lemma 5.4. *Cases (d), (f) and (g) do not occur.*

Proof. Let I_S be the ideal sheaf of $S \subset \mathbb{P}^4$. Generally, the global sections of $I_S^m(n)$ and $\mathcal{O}_{P'}(M)$ are bijective canonically. So we have

$$(5.3) \quad h^0(\mathbb{P}^4, I_S^m(n)) = h^0(P', M) = 5$$

by equation (1.4).

We prove the lemma case by case. In each case, we prove by contradiction in the following situations

$$h^0(\mathbb{P}^4, I_S(4)) = 0, = 1 \text{ and } \geq 2.$$

Assume Case (d) holds.

Suppose $h^0(\mathbb{P}^4, I_S(4)) = 0$. Consider the surjective map

$$(5.4) \quad \bigoplus_{k_1+k_2=9} H^0(\mathbb{P}^4, I_S(k_1)) \otimes H^0(\mathbb{P}^4, I_S(k_2)) \twoheadrightarrow H^0(\mathbb{P}^4, I_S^2(9)).$$

By hypothesis $h^0(\mathbb{P}^4, I_S(k)) = 0$ for all $k \leq 4$. Since $k_1 + k_2 = 9$ implies $k_1 \leq 4$ or $k_2 \leq 4$, the left hand side of (5.4) vanishes. Thus $h^0(\mathbb{P}^4, I_S^2(9)) = 0$, contradicting to (5.3).

Suppose $h^0(\mathbb{P}^4, I_S(4)) = 1$. Let $A \in H^0(\mathbb{P}^4, I_S(4))$ be a generator. This forces $h^0(\mathbb{P}^4, I_S(k)) = 0$ for all $k \leq 3$. It follows that $H^0(\mathbb{P}^4, I_S^2(8))$ is generated by A^2 . Let X_0, \dots, X_4 be a basis of one forms on \mathbb{P}^4 . Then (5.3) indicates that $A^2 X_0, \dots, A^2 X_4$ form a basis for $H^0(\mathbb{P}^4, I_S^2(9))$. As a result, the linear system $|I_S^2(9)|$ defines an automorphism of \mathbb{P}^4 instead of a Cremona transformation.

Suppose $h^0(\mathbb{P}^4, I_S(4)) \geq 2$. Let $A, B \in H^0(\mathbb{P}^4, I_S(4))$ be independent. Then A^2 and AB are independent in $H^0(\mathbb{P}^4, I_S^2(8))$. We claim that there exists i such that $A^2 X_i$ is not a linear combination of ABX_j , $j = 0, \dots, 4$. Suppose not, i.e. $A^2 X_i = ABL_i$ for some linear form L_i , $i = 0, \dots, 4$. Then we have $\frac{A}{B} = \frac{L_0}{X_0} = \frac{L_1}{X_1}$, which implies that $L_0 = \frac{X_0 L_1}{X_1}$, so X_1 divides L_1 . Therefore $\frac{A}{B} = \frac{L_1}{X_1}$ is a scalar, thus A and B are dependant, a contradiction. As a result, there exists i such that $A^2 X_i$ and ABX_0, \dots, ABX_4 form an independant subset of $H^0(\mathbb{P}^4, I_S^2(9))$. Thus $h^0(\mathbb{P}^4, I_S^2(9)) \geq 6 > 5$, a contradiction.

Assume Case (f) holds.

Suppose $h^0(\mathbb{P}^4, I_S(4)) = 0$. Then $h^0(\mathbb{P}^4, I_S(k)) = 0$ for all $k \leq 4$. Now we consider the map

$$(5.5) \quad \bigoplus_{k_1 + \dots + k_5 = 24} H^0(\mathbb{P}^4, I_S(k_1)) \otimes \dots \otimes H^0(\mathbb{P}^4, I_S(k_5)) \twoheadrightarrow H^0(\mathbb{P}^4, I_S^5(24)).$$

At least one $k_i \leq 4$, $i = 1, \dots, 5$, if their sum equals 24. Hence the left hand side of (5.5) vanishes. Thus $h^0(\mathbb{P}^4, I_S^5(24)) = 0 \neq 5$.

Suppose $h^0(\mathbb{P}^4, I_S(4)) \geq 1$. Let $A \in H^0(\mathbb{P}^4, I_S(4))$ be a nonzero element. Then $A^5 \in H^0(\mathbb{P}^4, I_S^5(20))$. Multiplication by A^5 defines an injection

$$\cdot A^5 : H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(4)) \hookrightarrow H^0(\mathbb{P}^4, I_S^5(24)).$$

Thus $h^0(\mathbb{P}^4, I_S^5(24)) \geq \binom{8}{4} = 70 > 5$, a contradiction.

The elimination of Case (g) is similar to Case (f). In Case (g), we use the surjection

$$\bigoplus_{k_1 + \dots + k_{10} = 49} H^0(\mathbb{P}^4, I_S(k_1)) \otimes \dots \otimes H^0(\mathbb{P}^4, I_S(k_{10})) \twoheadrightarrow H^0(\mathbb{P}^4, I_S^{10}(49))$$

to rule out the situation $h^0(\mathbb{P}^4, I_S(4)) = 0$. If $H^0(\mathbb{P}^4, I_S(4))$ contains $A \neq 0$, then the multiplication of A^{10} with 9-forms produces $\binom{13}{4} = 715$ independent elements in $H^0(\mathbb{P}^4, I_S^{10}(49))$, which is not allowed. \square

Lemma 5.5. *In cases (a) and (b) we have*

	(a) (3, 1, 2)	(b) (4, 1, 4)
d	≤ 8	≤ 15
$K_{\Sigma}C$	$4d - 25$	$7d - 60$
K_{Σ}^2	$d^2 - 11d + 4\delta + 30$	$d^2 + d + 4\delta - 105$
$c_2(\Sigma)$	$19d - 95 + 6\delta$	$46d - 405 + 6\delta$
$12\chi(O_{\Sigma})$	$d^2 + 8d - 65 + 10\delta$	$d^2 + 47d - 510 + 10\delta$
$g(C)$	$\frac{5d-23}{2}$	$4d - 29$

The invariants d and δ satisfy $(d-5)^2 = 2\delta$ in Case (a) and $(d-10)(d-15) = 2\delta$ in Case (b) respectively.

Proof. In order to compute the invariants in the list, we first use (5.1) to express $K_{\Sigma}C$ in d with given n , m and ξ . Then (5.2) (resp. (5.2')) allows us to express $c_2(\Sigma)$ (resp. K_{Σ}^2) in d and δ . We compute $12\chi(O_{\Sigma})$ and $g(C)$ by Noether's formula and genus formula, respectively. The upper bound for d comes from the inequality $d < (n/m)^2$ which holds generally [CK89, Formulae 0.3 (v)].

We have $h^0(\mathbb{P}^4, I_S(n)) = h^0(P', M) = 5$ by (1.4). On the other hand, $h^1(\mathbb{P}^4, I_S(n)) = 0$ by [Dol10, Prop. 1.3.3]. Hence

$$h^0(\mathbb{P}^4, I_S(n)) = \chi(\mathbb{P}^4, I_S(n)) = \chi(P, I_{S'}(n))$$

where second equality follows from the functoriality of Euler characteristic. The short exact sequence

$$0 \rightarrow I_{S'}(nL - 2\Sigma_i E_i) \rightarrow \mathcal{O}_P(nL - 2\Sigma_i E_i) \rightarrow \mathcal{O}_{S'}(nC - 2\Sigma_i(Q'_i + Q''_i)) \rightarrow 0$$

implies that

$$\chi(P, I_{S'}(n)) = \chi(P, nL - 2\Sigma_i E_i) - \chi(S', nC - 2\Sigma_i(Q'_i + Q''_i)).$$

$\chi(P, nL - 2\Sigma_i E_i)$ counts the dimension of degree n polynomials singular along Δ , so

$$\chi(P, nL - 2\Sigma_i E_i) = \binom{n+4}{4} - 5\delta.$$

By the previous computations and the Riemann-Roch formula, we have

$$\chi(S', nC - 2\Sigma_i(Q'_i + Q''_i)) = \begin{cases} \frac{1}{12}(d^2 - 10d + 385 - 62\delta) & \text{for (a)} \\ \frac{1}{12}(d^2 - 25d + 930 - 62\delta) & \text{for (b).} \end{cases}$$

Thus

$$\chi(P, I_{S'}(n)) = \begin{cases} -\frac{1}{12}(d^2 - 10d - 35 - 2\delta) & \text{for (a)} \\ -\frac{1}{12}(d^2 - 25d + 90 - 2\delta) & \text{for (b).} \end{cases}$$

Then the two equations are obtained by setting $\chi(P, I_{S'}(n)) = 5$. \square

Lemma 5.6. *Case (a) does not occur.*

Proof. Assume (a) is satisfied. Then the same argument as in [CK89, Theorem 3.3] implies that $d = 5$. By Lemma 5.5 we have $\delta = 0$. \square

5.4. Geometric analysis of the remaining case. To complete the proof of Theorem 5.1, it remains to analyze the geometry of the last possible case:

Proposition 5.7. *If S has a double point then it can only be the image of a K3 surface $R \subset \mathbb{P}^7$ of degree 12 projected from three points on R .*

Proof. We know from the previous section that we are in Case (b).

We apply Lemma 5.5 several times. We have $d \leq 15$ and $g(C) = 4d - 29 \geq 0$ so $8 \leq d \leq 15$. Then the equation $(d - 10)(d - 15) = 2\delta$ follows, and our hypothesis $\delta > 0$ forces $d = 8$ or 9 . If $d = 8$ then $\delta = 7$, and then $\chi(\mathcal{O}_\Sigma) = 0$. This can't happen since Σ is projective. Suppose $d = 9$. Let R be the minimal model of Σ with canonical divisor K_R . So there is a morphism $\epsilon : \Sigma \rightarrow R$ contracting the (-1) -curves $F_1, \dots, F_r \subset \Sigma$ and $K_\Sigma = \epsilon^*K_R + \sum_i F_i$. Note that $\delta = 3$, so $K_\Sigma C = 3$, $K_\Sigma^2 = -3$ and $\chi(\mathcal{O}_\Sigma) = 2$.

The remainder of argument is organized by the Kodaira dimension κ of R .

If $\kappa = 2$, then R is of general type. By [BHPVdV04, Thm. VII.2.2], we have $K_R^2 > 0$, which guarantees that R is projective by [BHPVdV04, Thm. IV.6.2]. Let H be a very ample divisor on R such that $C = \epsilon^*H - \sum_{i=1}^r \alpha_i F_i$ for some $\alpha_i \geq 1$, $i = 1, \dots, r$. We have $K_R H > 0$ by [BHPVdV04, Cor. VII.2.3]. It follows that $3 = K_\Sigma C = (\epsilon^*K_R + \sum_i F_i)(\epsilon^*H - \sum_i \alpha_i F_i) = K_R H + \sum_{i \geq 1} \alpha_i > \sum_{i \geq 1} \alpha_i \geq r \geq 0$. Therefore $r = 0, 1$ or 2 , which implies that $K_R^2 = -3, -2$ or -1 , respectively. This contradicts to $K_R^2 > 0$.

If $\kappa = 1$, then R is an elliptic surface with $K_R^2 = r - 3 = 0$ by [Bea96, Prop. IX.2]. So Σ is the blowup of R at three points. Let $R \rightarrow B$ be the elliptic fibration. The projectivity of Σ confirms that R admits a section because the pullbacks of the fiber components of R and the exceptional divisors are unable to generate an ample divisor on Σ . It follows that R and B share the same first Betti number [SS10, 6.10], so $b_1(B) = b_1(R) = b_1(\Sigma) = 0$. It follows that $B \simeq \mathbb{P}^1$ and thus $\chi(\mathcal{O}_R) > 2$ by [SS10, 4.10], which contradicts to $\chi(\mathcal{O}_R) = \chi(\mathcal{O}_\Sigma) = 2$.

If $\kappa = 0$, then R is a K3 surface. Σ is the blowup of R at three points because $r = K_R^2 + 3 = 3$. Let H be a very ample divisor on R such that $C = \epsilon^*H - \sum_{i=1}^r \alpha_i F_i$ for some $\alpha_i \geq 1$, $i = 1, \dots, 3$. We have $3 = K_\Sigma C = (\sum_i F_i)(\epsilon^*H - \sum_i \alpha_i F_i) = \alpha_1 + \alpha_2 + \alpha_3$, which implies $\alpha_i = 1, \forall i$. Thus $H^2 = C^2 - (\sum_i F_i)^2 = 9 + 3 = 12$. Therefore R is a degree 12 K3 surface in \mathbb{P}^7 and the birational map $R \dashrightarrow \Sigma \rightarrow S$ can be realized as the projection from three points on R .

Finally, $\kappa = -\infty$ implies $1 = \chi(\mathcal{O}_R) = \chi(\mathcal{O}_\Sigma) = 2$, a contradiction. \square

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